



Mathematics: analysis and approaches

Higher level

Paper 3

9 May 2023

Zone A afternoon | Zone B morning | Zone C afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a .

In this question you may either use the change of logarithm base formula $\log_a x = \frac{\ln x}{\ln a}$ or a graphic display calculator "logarithm to any base feature".

The function f is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+, a \neq 1.$$

- (a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero x -axis intercepts.

[4]

(This question continues on the following page)

(Question 1 continued)

In parts (b) and (c), consider the case where $a = e$. Note that $\ln x \equiv \log_e x$.

- (b) Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a local minimum. [5]
- (c) Hence deduce that $x > \ln x$. [1]
- (d) There exist values of a for which the graph of $y = \log_a x$ and the line $y = x$ do have intersection points. The following table gives three intervals for the value of a .

Interval	Number of intersection points
$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a , write down the values of p , q and r . [4]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

- (e) Find the exact coordinates of P and the exact value of a . [8]
- (f) Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have
- (i) two intersection points; [1]
- (ii) no intersection points. [1]

2. [Maximum mark: 31]

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where $m, c \in \mathbb{R}$ and $m, c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c , in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

(a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Consider $L(x) = mx + c$.

(b) (i) Show that $r = -\frac{c}{m}$. [1]

(ii) Given that $L(x)$ is an AS-linear function, show that $L(x) = mx - \frac{m^2}{m+2}$. [4]

(iii) State any further restrictions on the value of m . [1]

There are only three **integer** sets of values of m, r and c , that form an AS-linear function. One of these is $L(x) = -x - 1$.

(c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c . [3]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}, a \neq 0$ and $b, c \in \mathbb{R}$.

Let $r_1, r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

(d) Write down an expression for

(i) the sum of roots, $r_1 + r_2$, in terms of a and b . [1]

(ii) the product of roots, $r_1 r_2$, in terms of a and c . [1]

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(Question 2 continued)

If a, r_1, b, r_2 and c , in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

- (e) Given that $Q(x)$ is an AS-quadratic function,
- (i) write down an expression for $r_2 - r_1$ in terms of a and b ; [1]
 - (ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \frac{a^2 - ab - b}{2a}$; [2]
 - (iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\frac{1}{2}$. [3]

Consider the case where $b = 0$.

- (f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Now consider the case where $a = -\frac{1}{2}$.

- (g) (i) Find an expression for r_1 in terms of b . [2]
- (ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.
- Give your answers in the form $\frac{-p \pm q\sqrt{s}}{2}$ where $p, q, s \in \mathbb{Z}^+$. [5]
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